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Propagator for the Anisotropic Three-Dimensional Charged Harmonic Oscillator in A Constant Magnetic Field Using the Schwinger Action Principle

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The propagator for the anisotropic three-dimensional charged harmonic oscillator in the presence of a constant external magnetic field is calculated using the Schwinger action principle.

1. INTRODUCTION

Recently there has been a renewal in the interest of calculating exact propagators for quadratical systems in quantum mechanics. The emphasis now is upon systems whose Lagrangians exhibit a nonlocal time dependence in the coordinates which are referred to as systems with memory kernel (Khandekar et al., 1983a, 1983b; Brosens and Devreese, 1984; Cheng, 1984; Castrigiano and Kokiantonis, 1984). Except for some specific methods of calculation, the most popular way of deriving the propagator is via the Feynman path integral formulation of quantum mechanics (Feynman and Hibbs, 1965). In practice this approach is severely restricted because the art of calculating functional integrals is at its very beginning and only a few of them can be readily evaluated.

A most dramatic example of this situation shows up in a recent calculation (Cheng, 1984) of the propagator corresponding to a three-dimensional anisotropic charged harmonic oscillator in the presence of a constant external magnetic field. The author reduces the problem to the calculation of certain one-dimensional path integral related to the problem of an harmonic oscillator with generalized memory driven by an external time dependent force. Apparently such path integral cannot be calculated exactly at the

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present time and the author is only able to evaluate the particular case of zero frequency in the x direction. This limit does not even include the calculation for the propagator in the isotropic case made by Papadopoulos some years ago (Papadopoulos, 1971).

In this paper we present the calculation of the propagator for the anisotropic three-dimensional charged harmonic oscillator in a constant magnetic field using the Schwinger action principle. As far as we know this propagator has not been calculated previously.[2] Even though the basic ideas underlying this principle are well known we feel that the simplicity it provides in the calculation of propagators associated with quadratic systems is not fully appreciated. In general the calculation proceeds along the same lines as in the well-known case of the one-dimensional harmonic oscillator with constant frequency. The only surprises and complications that arise are only algebraic in nature and stem mainly from the solution of the operator equations of motion. Nevertheless, because we are dealing with quadratical systems, such equations are linear and their solution presents no more difficulties than in the classical problem.

In the following we present the basic features of the Schwinger action principle that we will need for our calculation and refer the reader to the original source (Schwinger, 1970) and to a recent discussion with applications (Urrutia and Hernández, 1984) for more details. The Schwinger action principle in quantum mechanics provides a differential characterization of the propagator in terms of the variations of the Lagrangian operator. Once the operator equations of motion for the dynamical variables are used the action principle reads ($\hbar = 1$)

$$\delta\langle x'', t''|x', t'\rangle = i\langle x'', t''|\mathscr{G}(t'') - \mathscr{G}(t')|x', t'\rangle \tag{1}$$

where

$$\mathscr{G}(t) = p(t)\delta x(t) - H(t)\delta t(t) \tag{2}$$

is the generator of time (δt) and coordinate (δx) displacements written in terms of the momentum $[p(t)]$ and Hamiltonian $[H(t)]$ operators. The generalization of (1) and (2) to more degrees of freedom is obvious. Let us remark that throughout this work we will only deal with dynamical operators of the first kind whose variations are just real numbers. Equation (1) must be subsequently integrated with respect to $\delta x''$, $\delta x'$, $\delta t''$, $\delta t'$ after the right-hand side matrix element is evaluated. In order to perform this evaluation the equations of motion for all the operators involved must be solved in terms of the operator boundary conditions $x(t'')$ and $x(t')$ whose action

$$\begin{aligned} x(t'')|x'', t''\rangle &= x''|x'', t''\rangle \\ x(t')|x', t'\rangle &= x'|x', t'\rangle \end{aligned} \tag{3}$$

[2]Please see the Added Comment at the end of the paper.

is immediate. This can always be done for quadratical systems (linear equations of motion) where the problem really reduces to solving the classical equations of motion with appropriate boundary conditions. The calculation of the part of the matrix element in (1) coming from the quadratic Hamiltonian will require the knowledge of the commutator $[x(t'), x(t'')]$ in order that each operator can be moved in front of the corresponding eigenvector to make use of (3). The contribution arising from the momenta is trivial to calculate because these operators are linear in $x(t'')$ and $x(t')$. Once the above steps are taken, (1) reduces to a purely numerical relation which can now be readily integrated. This task is simplified because the integrability conditions are automatically satisfied by virtue of the operator equations of motion (Urrutia and Hernández, 1984). Thus we perform first the coordinate integrations generated by the momenta matrix elements in (1) which will lead to terms quadratic in the end point coordinates. These terms must be recovered as the result of the time integrations in the matrix elements arising from the Hamiltonian in virtue of the integrability conditions. Because of this we only have to pay attention to those time integrations which are not quadratical in the coordinates and which arise from the use of the commutator referred to above.

In Section 2 we formulate the problem and discuss some related symmetry properties which are useful in solving the equations of motion. Section 3 contains the calculation of the propagator. Here we have proceeded in steps of increasing complexity starting from the case of zero harmonic potential, then going to the isotropic situation and finally to the general case. Of course this is not strictly necessary but it is very helpful in suggesting the convenient starting points for each more complicated case.

2. FORMULATION OF THE PROBLEM

We consider a three-dimensional charged harmonic oscillator, in the presence of a constant magnetic field, described by the Hamiltonian

$$H=\frac{1}{2m}\left[\left(p_x+\frac{m\omega}{2}y\right)^2+\left(p_y-\frac{m\omega}{2}x\right)^2+p_z^2\right]$$
$$+\frac{m}{2}(\Omega_1^2x^2+\Omega_2^2y^2+\Omega_3^2z^2) \tag{4}$$

The coordinate system is chosen in such a way that the constant magnetic field points in the z direction and $\omega=qB/mc$ is the associated Lamor frequency. The Hamiltonian (4) corresponds to the Lagrangian used by Cheng (Cheng, 1984). At any time t, the operators x, y, z, p_x, p_y, p_z satisfy the usual commutation relations which we do not bother to write.

We are interested in calculating the propagator $\langle x'', y'', z'', t''|x', y', z', t'\rangle$ for such system. The first obvious property we notice is that the problem is really two-dimensional because the z-motion decouples. In fact we have

$$\langle x'', y'', z'', t''|x', y', z', t'\rangle = \langle x'', y'', t''|x', y', t'\rangle\langle z'', t''|z', t'\rangle \tag{5}$$

where $\langle z'', t''|z', t'\rangle$ is the well-known propagator of a one-dimensional harmonic oscillator with mass m and frequency Ω_3. Thus we only need to be concerned with the x–y propagator.

The corresponding equations of motion obtained from the general relation $\dot{A} = i[A, H]$ are

$$\begin{aligned} \dot{x} &= \frac{1}{m}\left(p_x + \frac{m\omega}{2} y\right) \\ \dot{y} &= \frac{1}{m}\left(p_y - \frac{m\omega}{2} x\right) \\ \dot{p}_x &= \frac{\omega}{2}\left(p_y - \frac{m\omega}{2} x\right) - m\Omega_1^2 x \\ \dot{p}_y &= -\frac{\omega}{2}\left(p_x + \frac{m\omega}{2} y\right) - m\Omega_2^2 y \end{aligned} \tag{6}$$

As usual, the momenta p_x and p_y can be eliminated after taking an extra time derivative in the first two equations and then substituting $\dot{p}_x$ and $\dot{p}_y$ in the last ones. This yields the coupled system

$$\begin{aligned} \ddot{x} + \Omega_1^2 x &= \omega\dot{y} \\ \ddot{y} + \Omega_2^2 y &= -\omega\dot{x} \end{aligned} \tag{7}$$

whose appropriate solution constitutes the main difficulty of the method. We need to solve equations (7) introducing as boundary conditions the operators $x(t'')$, $y(t'')$ and $x(t')$, $y(t')$ with eigenvectors $|x'', y'', t''\rangle$ and $|x', y', t'\rangle$, respectively. Because the system (7) is linear its general solution will be a superposition of independent solutions associated with each of the four boundary conditions. Let us consider first the pair of solutions related to the boundary condition $x(t')$. Then we must have

$$\begin{aligned} x(t) &= x(t')F(t; t'', t') \\ y(t) &= x(t')G(t; t'', t') \end{aligned} \tag{8}$$

where the numerical functions F and G satisfy the coupled equations

$$\begin{aligned} \ddot{F} + \Omega_1^2 F &= \omega\dot{G} \\ \ddot{G} + \Omega_2^2 G &= -\omega\dot{F} \end{aligned} \tag{9}$$

with boundary conditions

$$F(t'; t'', t') = 1, \qquad F(t''; t'', t') = 0$$
$$G(t'; t'', t') = 0, \qquad G(t''; t'', t') = 0 \tag{10}$$

The system (9) can be decoupled in the usual manner in such a way that F and G satisfy the quartic equation

$$\ddddot{z} + (\Omega_1^2 + \Omega_2^2 + \omega^2)\ddot{z} + +\Omega_1^2\Omega_2^2 z = 0 \tag{11}$$

whose linearly independent solutions are given by cos $W_\pm t$; sin $W_\pm t$. The four roots $\pm W_\pm$ of the algebraic equation associated to (11) can be written as

$$W_+ = \tfrac{1}{2}(\omega_1 + \omega_2)$$
$$W_- = \tfrac{1}{2}(\omega_1 - \omega_2) \tag{12}$$

in terms of the positive numbers

$$\omega_1 = [(\Omega_1 + \Omega_2)^2 + \omega^2]^{1/2}$$
$$\omega_2 = [(\Omega_1 - \Omega_2)^2 + \omega^2]^{1/2} \tag{13}$$

Of course we still have to pay attention to the coupled nature of F and G. For example, if we take

$$F = \alpha \cos(Wt + \phi) \tag{14}$$

with α and ϕ constants and W either W_+ or W_-, the corresponding G is given by

$$G = \alpha\left(\frac{\omega W}{W^2 - \Omega_1^2}\right) \sin(Wt + \phi) \tag{15}$$

by virtue of (9). Moreover the coupling factor g satisfies

$$g \equiv \frac{\omega W}{W^2 - \Omega_1^2} = \frac{W^2 - \Omega_2^2}{\omega W} \tag{16}$$

which is just another way of writing the algebraic equation associated to (11).

It is interesting to remark that once we have obtained the explicit forms for F and G satisfying (9) and (10) the independent solutions having to do with the remaining boundary conditions can be easily generated from (8) using symmetry arguments. In fact, $x(t'')$ plays the same role of $x(t')$ at the time $t = t''$. Thus

$$x(t) = x(t'')F(t; t', t'')$$
$$y(t) = x(t'')G(t; t', t'') \tag{17}$$

are the independent solutions related to $x(t'')$ which are obtained from (8) just by interchanging t' and t''.

In order to obtain the two remaining independent solutions we notice that the system (9) is invariant under the replacements

$$x \to y, \qquad y \to -x, \qquad \Omega_1 \rightleftarrows \Omega_2 \tag{18}$$

Thus

$$\begin{aligned} x(t) &= -y(t')\tilde{G}(t;\, t'', t') \\ y(t) &= y(t')\tilde{F}(t;\, t'', t') \end{aligned} \tag{19}$$

and

$$\begin{aligned} x(t) &= -y(t'')\tilde{G}(t;\, t', t'') \\ y(t) &= y(t'')\tilde{F}(t;\, t', t'') \end{aligned} \tag{20}$$

are the independent solutions corresponding to the operator boundary conditions $y(t')$ and $y(t'')$, respectively. Here $\tilde{F}$ and $\tilde{G}$ are obtained from F and G after making the changes $\Omega_1 \rightleftarrows \Omega_2$. Notice that this transformation leaves ω_1 and ω_2 invariant as it is apparent from (13).

The complete solution of our system (9) is then the sum of the corresponding equations (8), (17), (19), and (20).

3. EVALUATION OF THE PROPAGATOR

In this section we calculate the propagator for our system. Even though it is not strictly necessary, we proceed in steps of increasing complexity starting from the case $\Omega_1 = \Omega_2 = 0$, then considering the situation $\Omega_1 = \Omega_2 = \Omega$ and finally dealing with the general case. We do this mainly because the simple cases contain very useful suggestions regarding the structure of the more complicated ones, thus allowing us to minimize the computational difficulties. Besides, the first two cases are already calculated in the literature which permit us to verify the correctness of our calculation.

3.1. The Case $\Omega_1 = \Omega_2 = 0$

This situation corresponds to the case where the charged particle does not feel any harmonic force in the x-y plane. Here we have $\omega_1 = \omega_2 = \omega$, $W_+ = \omega$, $W_- = 0$ and the functions F and G are

$$\begin{aligned} F(t;\, t'', t') &= \frac{1}{\sin(\omega T/2)} \sin\frac{\omega(t''-t)}{2} \cos\frac{\omega(t-t')}{2} \\ G(t;\, t'', t') &= -\frac{1}{\sin(\omega T/2)} \sin\frac{\omega(t''-t)}{2} \sin\frac{\omega(t-t')}{2} \end{aligned} \tag{21}$$

where $T=t''-t'$. It can be readily seen that F and G satisfy equations (9) together with the boundary conditions (10). Following the symmetry considerations prescribed in the last section we can immediately write the general solution

$$x(t)=\frac{1}{\sin(\omega T/2)}\left\{x(t')\sin\frac{\omega(t''-t)}{2}\cos\frac{\omega(t-t')}{2}\right.$$
$$+x(t'')\cos\frac{\omega(t''-t)}{2}\sin\frac{\omega(t-t')}{2}$$
$$\left.+[y(t')-y(t'')]\sin\frac{\omega(t''-t)}{2}\sin\frac{\omega(t-t')}{2}\right\} \tag{22}$$

$$y(t)=\frac{1}{\sin(\omega T/2)}\left\{[x(t'')-x(t')]\sin\frac{\omega(t''-t)}{2}\sin\frac{\omega(t-t')}{2}\right.$$
$$+y(t'')\cos\frac{\omega(t''-t)}{2}\sin\frac{\omega(t-t')}{2}$$
$$\left.+y(t')\sin\frac{\omega(t''-t)}{2}\cos\frac{\omega(t-t')}{2}\right\} \tag{23}$$

For the application of the action principle we will need the momenta at the end points which are

$$\begin{aligned}
p_x(t'')&=\frac{m\omega}{2}\left\{[x(t'')-x(t')]\cot\frac{\omega T}{2}-y(t')\right\}\\
p_x(t')&=\frac{m\omega}{2}\left\{[x(t'')-x(t')]\cot\frac{\omega T}{2}-y(t'')\right\}\\
p_y(t'')&=\frac{m\omega}{2}\left\{x(t')+[y(t'')-y(t')]\cot\frac{\omega T}{2}\right\}\\
p_y(t')&=\frac{m\omega}{2}\left\{x(t'')+[y(t'')-y(t')]\cot\frac{\omega T}{2}\right\}
\end{aligned} \tag{24}$$

From these expressions we can easily calculate the corresponding velocities according to equations (6).

Now we are in position to obtain the partial variations of the transformation function with respect to the end point variables according to the obvious generalization of (1) and (2). The variation with respect to the end-point coordinates can be readily evaluated because the momenta are linear in the position operators. The answer is

$$i\langle x'', y'', t''|x', y', t'\rangle \frac{m\omega}{2}\left[\delta(x'y''-x''y') + \frac{1}{2}\cot\frac{\omega T}{2}\delta((x''-x')^2+(y''-y')^2)\right] \tag{25}$$

The contribution coming from the variation of the end-points time variables is related to the Hamiltonian of the system which is quadratical in the position operators. The integrability conditions referred to in the Introduction assure us that the integration with respect to T of such quadratical terms will reproduce the piece $\frac{1}{2}\cot(\omega T/2)[(x''-x')^2+(y''-y')^2]$ arising from (25). In the process of calculating the matrix element of the quadratic combinations of position operators we will need the appropriate commutators in order to move each operator in front of its eigenvector. Such nonzero commutators are

$$\begin{aligned}
[x(t'), x(t'')] &= [y(t'), y(t'')] = \frac{i}{m\omega}\sin\omega T \\
[x(t'), y(t'')] &= -[y(t'), x(t'')] = -\frac{2i}{m\omega}\sin^2\frac{\omega T}{2}
\end{aligned} \tag{26}$$

and they will generate the following purely time dependent contribution to the total variation of the transformation function

$$-\langle x'', y'', t''|x', y', t'\rangle\delta\left(\ln\sin\frac{\omega T}{2}\right) \tag{27}$$

Putting together the results (25) and (27) we arrive at the final result

$$\langle x'', y'', t''|x', y', t'\rangle = \frac{A}{\sin(\omega T/2)}\exp\frac{im\omega}{2}\left\{(x'y''-x''y') + \tfrac{1}{2}[(x''-x')^2+(y''-y')^2]\cot\frac{\omega T}{2}\right\} \tag{28}$$

where the normalization constant A is independent of the end point variables and is fixed by comparing (28) with the well-known two-dimensional free particle limit $T\to 0$. We obtain

$$A = \frac{m\omega}{4\pi i} \tag{29}$$

and the result (28) coincides with previous calculations [Kennard (1927) and Papadopoulos (1971), among others].

3.2. The Case $\Omega_1 = \Omega_2 = \Omega$

This situation corresponds to a trivial extension of the calculation made by Papadopoulos (Papadopoulous, 1971) for the propagator corresponding to the charged isotropic harmonic oscillator in the constant external magnetic field. Here $\omega_1 = (4\Omega^2 + \omega^2)^{1/2}$, $\omega_2 = \omega$, and the coupling factors $g_\pm$ are still equal to 1. The functions F and G are now

$$
\begin{aligned}
F(t;\, t'', t') &= \frac{1}{\sin(\omega_1 T/2)}\left[\sin\frac{\omega_1(t''-t)}{2}\cos\frac{\omega_2(t-t')}{2}\right] \\
G(t;\, t'', t') &= -\frac{1}{\sin(\omega_1 T/2)}\left[\sin\frac{\omega_1(t''-t)}{2}\sin\frac{\omega_2(t-t')}{2}\right]
\end{aligned}
\tag{30}
$$

which reproduce equation (21) in the limit $\Omega \to 0$. Of course we can verify that equations (30) indeed satisfy the coupled system (9) with $\Omega_1 = \Omega_2 = \Omega$ together with the boundary conditions (10). The complete solution can again be generated using the symmetry properties of the previous section and the final form, which we do not write explicitly here, is completely analogous to equations (22) and (23) with the appropriate identification of ω_1 and ω_2. Once more we will need the momenta at the initial and final times

$$
\begin{aligned}
p_x(t'') &= \frac{m\omega_1}{2\sin(\omega_1 T/2)}\left[x(t'')\cos\frac{\omega_1 T}{2} - x(t')\cos\frac{\omega_2 T}{2} - y(t')\sin\frac{\omega_2 T}{2}\right] \\
p_x(t') &= \frac{m\omega_1}{2\sin(\omega_1 T/2)}\left[x(t'')\cos\frac{\omega_2 T}{2} - x(t')\cos\frac{\omega_1 T}{2} - y(t'')\sin\frac{\omega_2 T}{2}\right] \\
p_y(t'') &= \frac{m\omega_1}{2\sin(\omega_1 T/2)}\left[x(t')\sin\frac{\omega_2 T}{2} + y(t'')\cos\frac{\omega_1 T}{2} - y(t')\cos\frac{\omega_2 T}{2}\right] \\
p_y(t') &= \frac{m\omega_1}{2\sin(\omega_1 T/2)}\left[x(t'')\sin\frac{\omega_2 T}{2} + y(t'')\cos\frac{\omega_2 T}{2} - y(t')\cos\frac{\omega_1 T}{2}\right]
\end{aligned}
\tag{31}
$$

The variation of the propagator with respect to the end-point coordinates can now be easily calculated and the result is

$$
\begin{aligned}
i\langle x'', y'', t'' | x', y', t'\rangle \frac{m\omega_1}{2}\Bigg[&\frac{1}{2}\cot\frac{\omega_1 T}{2}\,\delta(x''^2 + y''^2 + x'^2 + y'^2) \\
&- \frac{\cos(\omega_2 T/2)}{\sin(\omega_1 T/2)}\,\delta(x'x'' + y'y'') + \frac{\sin(\omega_2 T/2)}{\sin(\omega_1 T/2)}\,\delta(x'y'' - y'x'')\Bigg]
\end{aligned}
\tag{32}
$$

This result correctly reproduces the equation (25) in the limit $\Omega \to 0$. The nonzero commutators of the end-point operators are

$$\begin{aligned}
[x(t'), x(t'')] &= [y(t'), y(t'')] = \frac{2i}{m\omega_1} \sin\frac{\omega_1 T}{2} \cos\frac{\omega_2 T}{2} \\
[x(t'), y(t'')] &= -[y|(t'), x(t'')] = -\frac{2i}{m\omega_1} \sin\frac{\omega_1 T}{2} \sin\frac{\omega_2 T}{2}
\end{aligned} \tag{33}$$

which are used in calculating the coordinate independent part of the elapsed time $T = t'' - t'$ variation of the propagator. This contribution is

$$-\langle x'', y'', t''|x', y', t'\rangle \delta\left(\ln \sin\frac{\omega_1 T}{2}\right) \tag{34}$$

Recalling once more that the integrability conditions are automatically satisfied by virtue of the operator equations of motion we finally obtain the result

$$\begin{aligned}
\langle x'', y'', t''|x', y', t'\rangle = \frac{A_1}{\sin(\omega_1 T/2)} \exp \frac{im\omega_1}{2\sin(\omega_1 T/2)} \\
\times \left[\sin\frac{\omega_2 T}{2}(x'y'' - x''y') - \cos\frac{\omega_2 T}{2}(x''x' + y''y') \right. \\
\left. + \frac{1}{2}\cos\frac{\omega_1 T}{2}(x''^2 + y''^2 + x'^2 + y'^2)\right]
\end{aligned} \tag{35}$$

where the normalization constant A_1 is

$$A_1 = \frac{m(4\Omega^2 + \omega^2)^{1/2}}{4\pi i}$$

The result (35) coincides with the one obtained by Papadopoulos (Papadopoulous, 1971), after the identifications $\Omega' \to \omega_1/2$ and $\omega \to \omega_2$ are made.

3.3. The General Case

In this section we discuss the general situation where the harmonic oscillator potential is completely anisotropic and which generalizes all previous cases. Here ω_1 and ω_2 take their general values given in (13) and the coupling factors written in (16) are not equal to 1. In order to find the appropriate F and G we start from the following ansatz, suggested by the previous forms,

$$\begin{aligned}
G(t; t'', t') = \alpha \sin\frac{\omega_1(t'' - t)}{2} \sin\frac{\omega_2(t - t')}{2} \\
+ \beta \sin\frac{\omega_2(t'' - t)}{2} \sin\frac{\omega_1(t - t')}{2}
\end{aligned} \tag{36}$$

where α and β are constants to be determined. Obviously, G given above satisfies the boundary conditions (10) and the quartic equation (11) because it can be easily rewritten as a sum of $\cos(W_{\pm}t+\phi)$. With the function G expressed in this form and using the coupling relation given by (14) and (15) we can immediately obtain the corresponding function F which we still want to write in a form similar to (21) and (30). The constants α and β are determined at this stage of the calculation by requiring F to satisfy the boundary condition (10). The corresponding equations are

$$\begin{aligned}\frac{\alpha}{2}(g_- - g_+)\sin\frac{\omega_1 T}{2} - \frac{\beta}{2}(g_+ + g_-)\sin\frac{\omega_2 T}{2} &= 1\\ \frac{\alpha}{2}(g_+ + g_-)\sin\frac{\omega_2 T}{2} - \frac{\beta}{2}(g_- - g_+)\sin\frac{\omega_1 T}{2} &= 0\end{aligned} \tag{37}$$

with $g_{\pm} = g(W_{\pm})$ according to the definition (16). Solving for α and β we obtain

$$\begin{aligned}\alpha &= -\frac{2\omega\Omega_1 p}{p^2 - q^2}\\ \beta &= \frac{q}{p}\alpha\end{aligned} \tag{38}$$

where we have introduced the notation

$$\begin{aligned}p &= \omega_2(\Omega_2 + \Omega_1)\sin\frac{\omega_1 T}{2}\\ q &= \omega_1(\Omega_2 - \Omega_1)\sin\frac{\omega_2 T}{2}\end{aligned} \tag{39}$$

In performing this calculation we have made use of the properties

$$\begin{aligned}g_+ + g_- &= \frac{\omega_1(\Omega_1 - \Omega_2)}{\omega\Omega_1}\\ g_+ - g_- &= \frac{\omega_2(\Omega_1 + \Omega_2)}{\omega\Omega_1}\end{aligned} \tag{40}$$

which are just consequence of the facts that W_+^2, W_-^2 are the roots of (16) together with the definitions (12).

Now we go back to the function F. When written in terms of the basic functions $\cos(W_{\pm}t+\phi)$ it will contain the products $\alpha g_{\pm}$, $\beta g_{\pm}$ of the parameters appearing in (38) with the coupling factors (40). We have chosen to eliminate $g_\pm$ in favor of α and β via the equations (37), which leads to the following expression for F:

$$\begin{aligned}F(t;\,t'',\,t') = \frac{1}{\alpha^2-\beta^2}\bigg\{ & \frac{\alpha^2}{\sin(\omega_1 T/2)}\sin\frac{\omega_1(t''-t)}{2}\cos\frac{\omega_2(t-t')}{2} \\ & -\frac{\beta^2}{\sin(\omega_2 T/2)}\sin\frac{\omega_2(t''-t)}{2}\cos\frac{\omega_1(t-t')}{2} \\ & +\alpha\beta\bigg[\frac{1}{\sin(\omega_2 T/2)}\cos\frac{\omega_1(t''-t)}{2}\sin\frac{\omega_2(t-t')}{2} \\ & -\frac{1}{\sin(\omega_1 T/2)}\cos\frac{\omega_2(t''-t)}{2}\sin\frac{\omega_1(t-t')}{2}\bigg]\bigg\}\end{aligned} \tag{41}$$

Let us notice that $q=\beta=0$ and $\alpha=-1/\sin(\omega_1 T/2)$ in the case $\Omega_1=\Omega_2=\Omega$. Thus equations (36) and (41) reduce to (30). From equation (36) and (41) the general solution can now be obtained in the usual manner. We will only write the expressions that are directly used in the rest of the calculation and which refer only to the end points. The momenta operators are given by

$$\begin{aligned} p_x(t'') &= M_1 x(t') + M_2 x(t'') - M_3 y(t') + M_4 y(t'') \\ p_x(t') &= -M_2 x(t') - M_1 x(t'') + M_4 y(t') - M_3 y(t'') \\ p_y(t'') &= M_3 x(t') + M_4 x(t'') + \tilde{M}_1 y(t') + \tilde{M}_2 y(t'') \\ p_y(t') &= M_4 x(t') + M_3 x(t'') - \tilde{M}_2 y(t') - \tilde{M}_1 y(t'') \end{aligned} \tag{42}$$

which are simply related to the corresponding velocity operators. The auxiliary functions M_i $(i=1,2,3,4)$ are

$$\begin{aligned} M_1 &= -\frac{\omega_1\omega_2}{p^2-q^2}\Omega_1\left(p\cos\frac{\omega_2 T}{2}+q\cos\frac{\omega_1 T}{2}\right) \\ M_2 &= \frac{\omega_1\omega_2}{p^2-q^2}\Omega_1\left(p\cos\frac{\omega_1 T}{2}+q\cos\frac{\omega_2 T}{2}\right) \\ M_3 &= \frac{2\omega\Omega_1\Omega_2}{p^2-q^2}\,\frac{pq}{\Omega_2^2-\Omega_1^2} \\ M_4 &= \frac{\omega}{2(p^2-q^2)}\left(\frac{\Omega_2-\Omega_1}{\Omega_2+\Omega_1}p^2-\frac{\Omega_2+\Omega_1}{\Omega_2-\Omega_1}q^2\right) \end{aligned} \tag{43}$$

The corresponding functions $\tilde{M}_i$ are obtained from M_i by changing $\Omega_1 \rightleftarrows \Omega_2$. Notice that $M_3=\tilde{M}_3$ and $M_4=-\tilde{M}_4$. In writing the equations (43) we have made extensive use of the relations (38).

The change of the transformation function with respect to the end-point coordinates, generated by the momentum operators, is now given by

$$i\langle x'', y'', t''|x', y', t'\rangle m\left[\frac{M_2}{2}\delta(x''^2+x'^2)+\frac{\tilde{M}_2}{2}\delta(y''^2+y'^2)+M_1\delta(x'x'')\right.$$
$$\left.+\tilde{M}_1\delta(y'y'')+M_4\delta(y''x''-y'x')+M_3\delta(x'y''-y'x'')\right] \tag{44}$$

which reduces to (32) in the limit $\Omega_1=\Omega_2=\Omega$.

The commutators employed in the evaluation of the change of the propagator coming from the Hamiltonian are

$$[x(t'), x(t'')] = -\frac{i}{m}\frac{\tilde{M}_1}{M}$$
$$[y(t'), y(t'')] = -\frac{i}{m}\frac{M_1}{M} \tag{45}$$
$$[y(t'), x(t'')] = -[x(t'), y(t'')] = \frac{i}{m}\frac{M_3}{M}$$

where $M = M_3^2+M_1\tilde{M}_1$ is given by

$$M = \frac{\omega_1^2\omega_2^2\Omega_1\Omega_2}{p^2-q^2} \tag{46}$$

The calculation is completed by giving the coordinate-independent contribution to the elapsed time variation of the propagator which is

$$-\langle x'', y'', t''|x', y', t'\rangle\delta(\ln(p^2-q^2)^{1/2}) \tag{47}$$

Collecting equations (44), (47) and recalling the integrability properties we obtain the final result:

$$\langle x'', y'', t''|x', y', t'\rangle = A_2(p^2-q^2)^{-1/2}\exp im[\tfrac{1}{2}M_2(x'^2+x''^2)+\tfrac{1}{2}\tilde{M}_2(y'^2+y''^2)$$
$$+M_1x'x''+\tilde{M}_1y'y''+M_4(y''x''-y'x')+M_3(x'y''-y'x'')] \tag{48}$$

where the normalization constant A_2 is

$$A_2 = \frac{m}{2\pi i}\omega_1\omega_2(\Omega_1\Omega_2)^{1/2} \tag{49}$$

The propagator (48) constitutes the main result of this paper and when substituted in equation (5) gives the general expression for the transformation function of a charged particle in an anisotropic oscillator well in the presence of an external magnetic field.

Added Comment. After our calculation was completed and this paper was written we became aware of the very recent work of Kokiantonis and Castrigiano (1985) where the same problem is solved. Nevertheless, we decided to submit our paper for publication mainly for two reasons:

(i) Our calculation constitutes an independent verification of the above-mentioned result since we use a method different from theirs.

(ii) We present a detailed account of the calculation where the laborious algebraic manipulations are minimized by the adequate use of symmetry considerations together with information obtained from simpler cases.

We have explicitly verified that our result (48) for the propagator coincides with the one obtained by Kokiantonis and Castrigiano (1985) after the following identifications between our paper and theirs are, respectively, made

$$\Omega_1 = \omega_x, \qquad \Omega_2 = \omega_y$$

$$W_\pm \equiv \tfrac{1}{2}(\omega_1 \pm \omega_2) = \lambda_\pm$$

$$p^2 - q^2 = \omega^2 \omega_x \omega_y D$$

$$M_1 = \frac{a_3}{2D}, \qquad M_2 = \frac{a_1}{D}, \qquad M_3 = \frac{a_5}{2D}, \qquad M_4 = \frac{a_6}{2D}$$

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